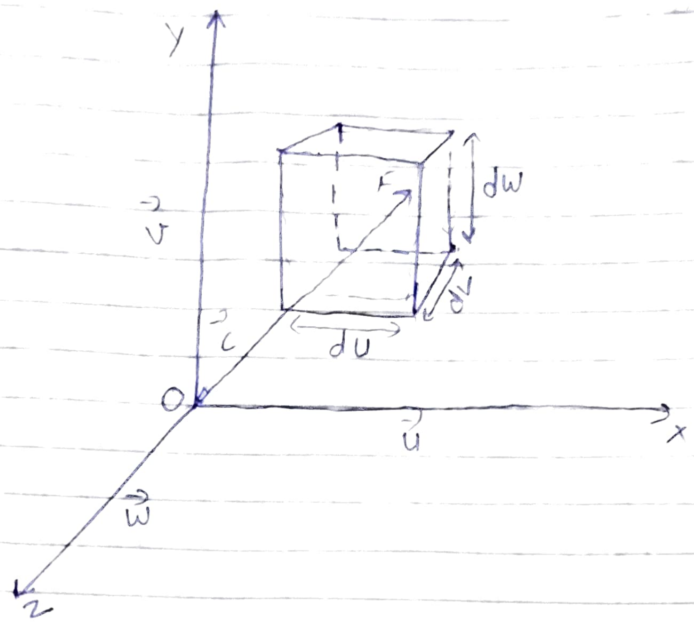
Derivation of maxwell's law of distribution of velocities



Let OX, OY and OZ are perpendicular co-ordinates axes with origin O

Z have components u, v and w along three axes

$$c^2 = u^2 + v^2 + w^2 \quad \text{--- (1)}$$

A point P is called the velocity point. $f(u)\,du$ represent the probability that a molecule selected at random has a velocity between u and $u+du$

Similarly $f(v)\,dv$ represent the probability that a molecule selected at random has a velocity between v and $v+dv$.

And $f(w)\,dw$ represent the probability that a molecule selected at random has a velocity between w and $w+dw$

Let a small volume element in velocity space is $du\,dv\,dw$, in which the molecules having component velocities between u + $u+du$, v and $v+dv$ and w and $w+dw$

Let N be the total no of molecules in given assembly of gas. In that dN be the no of gas molecules in a small volume element is

$$dN = N f(u) f(v) f(w) du dv dw \quad \text{—— (2)}$$

$$\frac{dN}{du\,dv\,dw} = N f(u) f(v) f(w)$$

or

$$\rho = N\phi(c^2) \quad \text{—— (3)}$$

where ρ represent density of gas molecules in velocity space

For fixed value of c

$$d\phi(c^2) = 0 \qquad \text{i.e. } d\rho = 0$$

$$\therefore d\{f(u) f(v) f(w)\} = 0 \quad \text{—— (4)}$$

Diff. eqn (4) w.r.t. u, v and w we get

$$f'(u) f(v) f(w) du + f'(v) f(u) f(w) dv + f'(w) f(u) f(v) dw = 0$$

Divide by $f(u) f(v) f(w)$ we get

$$\frac{f'(u)}{f(u)} du + \frac{f'(v)}{f(v)} dv + \frac{f'(w)}{f(w)} dw = 0 \quad \text{—— (5)}$$

For fixed value of c eqn (1) becomes

$$2u\,du + 2v\,dv + 2w\,dw = 0$$

$\therefore udu + vdv + wdw = 0$ — ( )

To multiply α to eq^n (6) and adding with eq^n (5) we get

$$\left[\frac{f'(u)}{f(u)} + \alpha u\right]du + \left[\frac{f'(v)}{f(v)} + \alpha v\right]dv + \left[\frac{f'(w)}{f(w)} + \alpha w\right]dw = 0$$

where α is arbitrary unknown function

As du, dv and dw are independent to each other, above eq^n becomes

$$\left[\frac{f'(u)}{f(u)} + \alpha u\right]du = 0 \quad \text{or} \quad \frac{f'(u)}{f(u)} + \alpha u = 0 \quad \text{— (a)}$$

Similary

$$\frac{f'(v)}{f(v)} + \alpha v = 0 \quad \text{and} \quad \text{— (b)}$$

$$\frac{f'(w)}{f(w)} + \alpha w = 0 \quad \text{— (c)}$$

Integrating eq^n (a), (b) and (c) we get as

$$\therefore f(u) = Ae^{-\alpha u^2/2}$$

Similary $f(v) = Ae^{-\alpha v^2/2}$

$$f(w) = Ae^{-\alpha w^2/2}$$

From eq^n (2)

eq^n (a) becomes

$$\frac{1}{f(u)}\frac{d}{du}f(u) = -\alpha u$$

$$\therefore \frac{d}{du}\left[\log f(u)\right] = -\alpha u$$

Integrating the terms as

$$\int d\left[\log f(u)\right] = \int \alpha u\, du$$

$$\log f(u) = -\alpha\frac{u^2}{2} \log$$

$$f(u) = Ae^{-\alpha u^2/2}$$

$$dN = NA^3 e^{-\alpha(u^2+v^2+w^2)/2}\, du\, dv\, dw$$

$$\frac{dN}{du\,dv\,dw} = \rho = NA^3 e^{-bc^2} \quad \text{---} (7)$$

where $b = \frac{\alpha}{2}$

It shows molecular density ρ is a function of c and it falls exponential with c.

As density ρ is constant within infinitesimal spherical shell, whose radii are in between c and $c+dc$

Volume of spherical shell is

$$du\,dv\,dw = 4\pi c^2 dc$$

$\therefore$ eq^n (7) becomes

$$dN = \rho 4\pi c^2 dc$$

$$= NA^3 e^{-bc^2} 4\pi c^2 dc$$

$$dN = 4\pi NA^3 e^{-bc^2} c^2 dc \quad \text{---} (8)$$

Here A and b are constants as

$$A = \sqrt{b/\pi} \quad \text{and} \quad b = \frac{m}{2KT}$$

eq^n (8) becomes

$$dN = 4\pi N \left(\frac{m}{2\pi KT}\right)^{3/2} \left(\frac{-m}{2KT}\right) c^2 dc \quad \text{---} (9)$$

As mean kinetic energy

$$E = \frac{1}{2} m c^2$$

Eq^n 9 becomes as

$$\frac{dN}{dc} = 4\pi N \left(\frac{m}{2\pi KT}\right)^{3/2} c^2 e^{(E/KT)} \quad \text{— A}$$

Eq^n A is known as maxwell's law of velocity distrubution.

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